The lower envelope of positive self-similar Markov processes.

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Abstract: We establish integral tests and laws of the iterated logarithm for the lower envelope of positive self-similar Markov processes at 0 and $+\infty$. Our proofs are based on the Lamperti representation and time reversal arguments. These results extend laws of the iterated logarithm for Bessel processes due to Dvoretsky and Erdös [11], Motoo [17] and Rivero [18].

Key words: Self-similar Markov process, Lévy process, Lamperti representation, last passage time, time reversal, integral test, law of the iterated logarithm.

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1 Introduction

A real self-similar Markov process $X^{(x)}$, starting from x is a càdlàg Markov process which fulfills a scaling property, i.e., there exists a constant $\alpha > 0$ such that for any $k > 0$,

$$\left(kX^{(x)}_{k^{-\alpha}t},\, t \geq 0\right) \stackrel{(d)}{=} (X^{(kx)}_t,\, t \geq 0). \tag{1.1}$$

Self-similar processes often arise in various parts of probability theory as limit of rescaled processes. Their properties have been studied by the early sixties under the impulse of Lamperti's work [14]. The Markov property added to self-similarity provides some interesting features as noted by Lamperti himself in [15] where the particular case of positive self-similar Markov processes is studied. These processes are involved for instance in branching processes and fragmentation theory. In this paper, we will consider *positive* self-similar Markov processes and refer to them as *pssMp*. Some

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particularly well known examples are transient Bessel processes, stable subordinators or more generally, stable Lévy processes conditioned to stay positive.

The aim of this work is to describe the lower envelope at 0 and at $+\infty$ of a large class of pssMp throughout integral tests and laws of the iterated logarithm (LIL for short). A crucial point in our arguments is the famous Lamperti representation of self-similar $\mathbb{R}_+$–valued Markov processes. This transformation enables us to construct the paths of any such process starting from $x > 0$, say $X^{(x)}$, from those of a Lévy process. More precisely, Lamperti [15] found the representation

$$X_t^{(x)} = x \exp \xi_{\tau(tx^{-\alpha})}, \quad 0 \le t \le x^{-\alpha} I(\xi)\,, \tag{1.2}$$

under $\mathbb{P}_x$, for $x > 0$, where

$$\tau_t = \inf\{s : I_s(\xi) \ge t\}\,, \quad I_s(\xi) = \int_0^s \exp \alpha \xi_u \, du\,, \quad I(\xi) = \lim_{t \to +\infty} I_t(\xi)\,,$$

and where ξ is a real Lévy process which is possibly killed at independent exponential time. Note that for $t < I(\xi)$, we have the equality $\tau_t = \int_0^t (X_s^{(x)})^{-\alpha} ds$, so that (1.2) is invertible and yields a one to one relation between the class of pssMp and the one of Lévy processes.

In this work, we consider pssMp's which drift towards $+\infty$, i.e. $\lim_{t\to+\infty} X_t^{(x)} = +\infty$, a.s. and which fulfills the Feller property on $[0, \infty)$, so that we may define the law of a pssMp, which we will call $X^{(0)}$, starting from 0 and with the same transition function as $X^{(x)}$, $x > 0$. Bertoin and Caballero [2] and Bertoin and Yor [3] proved that the family of processes $X^{(x)}$ converges, as $x \downarrow 0$, in the sense of finite dimensional distributions towards $X^{(0)}$ if and only if the underlying Lévy process ξ in the Lamperti's representation is such that

$$(H) \qquad \xi \text{ is non lattice and } \quad 0 < m \overset{\text{(def)}}{=} \mathbb{E}(\xi_1) \le \mathbb{E}(|\xi_1|) < +\infty\,.$$

As proved by Caballero and Chaumont in [5], the latter condition is also a NASC for the weak convergence of the family $(X^{(x)})$, $x \ge 0$ on the Skohorod's space of càdlàg trajectories. In the same article, the authors also provided a path construction of the process $X^{(0)}$. The entrance law of $X^{(0)}$ has been described in [2] and [3] as follows: for every $t > 0$ and for every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbb{E}\left(f\left(X_t^{(0)}\right)\right) = \frac{1}{m}\mathbb{E}\left(I(-\xi)^{-1} f(tI(-\xi)^{-1})\right)\,. \tag{1.3}$$

Several partial results on the lower envelope of $X^{(0)}$ have already been established before, the oldest of which being due to Dvoretsky and Erdös [11] and Motoo [17] who studied the special case of Bessel processes. More precisely, when $X^{(0)}$ is a Bessel process with dimension $\delta > 2$, we have the following integral test at 0: if f is an increasing function then

$$\mathbb{P}(X_t^{(0)} < f(t), \text{i.o., as } t \to 0) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as} \quad \int_{0+} \left(\frac{f(t)}{t}\right)^{\frac{\delta-2}{4}} \frac{dt}{t} \begin{cases} < \infty \\ = \infty \end{cases}.$$

The time inversion property of Bessel processes, induces the same integral test for the behaviour at $+\infty$ of $X^{(x)}$, $x \ge 0$. The test for Bessel processes is extended in the

section 4 of this paper to pssMp's such that the upper tail of the law of the exponential functional $I(-\xi)$ is regularly varying. Our integral test is then written in terms of the law of this exponential functional as shown in Theorem 3.

Without giving here an exhaustive list of the results which have been obtained in that direction, we may also cite Lamperti's own work [15] who used his representation to describe the asymptotic behaviour of a pssMp starting from $x > 0$ in terms of the underlying Lévy process. Some cases where the transition function of $X^{(x)}$ admits some special bounds have also been studied by Xiao [21].

The most recent result concerns increasing pssMp and is due to Rivero [18] who proved the following LIL: suppose that ξ is a subordinator whose Laplace exponent ϕ is regularly varying at infinity with index $\beta \in (0,1)$ and define the function $\varphi(t) = \phi(\log|\log t|)/\log|\log t|$, $t > e$, then

$$\liminf_{t\downarrow 0} \frac{X_t^{(0)}}{(t\varphi(t))^{1/\alpha}} = \alpha^{\beta/\alpha}(1-\beta)^{(1-\beta)/\alpha} \quad \text{and} \quad \liminf_{t\uparrow+\infty} \frac{X_t^{(0)}}{(t\varphi(t))^{1/\alpha}} = \alpha^{\beta/\alpha}(1-\beta)^{(1-\beta)/\alpha}, \ a.s.$$

In Section 5 of this paper, we extend Rivero's result to pssMp's such that the logarithm of the upper tail of the exponential functional $I(-\xi)$ is regularly varying at $+\infty$. In Theorem 4, we give a LIL for the process $X^{(0)}$ at 0 and for the processes $X^{(x)}$, $x \geq 0$ at $+\infty$. Then the lower envelope has an explicit form in terms of the tail of the law of $I(-\xi)$.

All the asymptotic results presented in Sections 4 and 5 are consequences of general integral tests which are stated and proved in Section 3 and which may actually be applied in other situations than our 'regular' and 'logregular' cases. If F_q denotes the tail of the law of the truncated exponential functionals $\int_0^{\hat{T}_{-q}} \exp -\xi_s\, ds$, $\hat{T}_{-q} = \inf\{t : \xi_s \leq -q\}$, $q \geq 0$, then we will show that in any case, the knowledge of asymptotic behaviour of F_q suffices to describe the lower envelope of the processes $X^{(x)}$, $x \geq 0$.

Section 2 is devoted to preliminary results. We give a path decomposition of the process $X^{(0)}$ up to its last passage time under a fixed level. This process, once reversed, corresponds to a pssMp whose associated Lévy process in the Lamperti transformation is $-\xi$. In particular, this allows us to get an expression of the last passage time process of $X^{(0)}$ in terms of $I(-\xi)$. The description of the last passage process is then used in section 3 for the study of the lower envelope of $X^{(x)}$.

2 Time reversal and last passage time of $X^{(0)}$

We consider processes defined on the space $\mathcal{D}$ of càdlàg trajectories on $[0,\infty)$, with real values. The space $\mathcal{D}$ is endowed with the Skorohod's topology and $\mathbb{P}$ will be our reference probability measure.

In all the rest of the paper, ξ will be a Lévy process satisfying condition (H). With no loss of generality, we will also suppose that $\alpha = 1$. Indeed, we see from (1.1) that if $X^{(x)}$, $x \geq 0$, is a pssMp with index $\alpha > 0$, then $(X^{(x)})^\alpha$ is a pssMp with index 1. Therefore, the integral tests and LIL established in the sequel can easily be interpreted for any $\alpha > 0$.

Let us define the family of positive self-similar Markov processes $\hat{X}^{(x)}$ whose Lamperti's representation is given by

$$\hat{X}^{(x)} = \left(x \exp \hat{\xi}_{\hat{\tau}(t/x)},\, 0 \le t \le xI(\hat{\xi})\right) , \quad x > 0 , \tag{2.4}$$

where $\hat{\xi} = -\xi$, $\hat{\tau}_t = \inf\{s : \int_0^s \exp \hat{\xi}_u \, du \ge t\}$, and $I(\hat{\xi}) = \int_0^\infty \exp \hat{\xi}_s \, ds$. We emphasize that the r.v. $xI(\hat{\xi})$, corresponds to the first time at which the process $\hat{X}^{(x)}$ hits 0, i.e.

$$xI(\hat{\xi}) = \inf\{t : \hat{X}_t^{(x)} = 0\} , \tag{2.5}$$

moreover, for each $x > 0$, the process $\hat{X}^{(x)}$ hits 0 continuously, i.e. $\hat{X}^{(x)}(xI(\hat{\xi})-) = 0$.

We now fix a decreasing sequence (x_n), $n \ge 1$ of positive real numbers which tends to 0 and we set

$$U(y) = \sup\{t : X_t^{(0)} \le y\} .$$

The aim of this section is to establish a path decomposition of the process $X^{(0)}$ reversed at time $U(x_1)$ in order to get a representation of this time in terms of the exponential functional $I(\hat{\xi})$, see Corollaries 2 and 3 below.

To simplify the notations, we set $\Gamma = X^{(0)}_{U(x_1)-}$ and we will denote by K the support of the law of Γ. We will see in Lemma 1 that actually $K = [0, x_1]$. For any process X that we consider here, we make the convention that $X_{0-} = X_0$.

Proposition 1 *The law of the process $\hat{X}^{(x)}$ is a regular version of the law of the process*

$$\hat{X} \overset{(def)}{=} (X^{(0)}_{(U(x_1)-t)-},\, 0 \le t \le U(x_1)) ,$$

conditionally on $\Gamma = x$, $x \in K$.

Proof: The result is a consequence of Nagasawa's theory of time reversal for Markov processes. First, it follows from Lemma 2 in [3] that the resolvent operators of $X^{(x)}$ and $\hat{X}^{(x)}$, $x > 0$ are in duality with respect to the Lebesgue measure. More specifically, for every $q \ge 0$, and measurable functions $f, g : (0, \infty) \to \mathbb{R}_+$, with

$$V^q f(x) \overset{(\mathrm{def})}{=} \mathbb{E}\left(\int_0^\infty e^{-qt} f(X_t^{(x)})\, dt\right), \quad \text{and} \quad \hat{V}^q f(x) \overset{(\mathrm{def})}{=} \mathbb{E}\left(\int_0^\zeta e^{-qt} f(\hat{X}_t^{(x)})\, dt\right),$$

we have

$$\int_0^\infty f(x) \hat{V}^q g(x)\, dx = \int_0^\infty g(x) V^q f(x)\, dx . \tag{2.6}$$

Let $p_t(dx)$ be the entrance law of $X^{(0)}$ at time t, then it follows from the scaling property that for any $t > 0$, $p_t(dx) = p_1(dx/t)$, hence $\int_0^\infty p_t(dx)\, dt = \int_0^\infty p_1(dy)/y\, dx$ for all $x > 0$, where from (1.3), $\int_0^\infty p_1(dy)/y\, dy = m^{-1}$. In other words, the resolvent measure of $\delta_{\{0\}}$ is proportional to the Lebesgue measure, i.e.:

$$m^{-1} \int_0^\infty f(x)\, dx = \mathbb{E}\left(\int_0^\infty f(X_t^{(0)})\, dt\right) . \tag{2.7}$$

Conditions of Nagasawa's theorem are satisfied as shown in (2.6) and (2.7), then it remains to apply this result to $U(x_1)$ which is a return time such that $\mathbb{P}(0 < U(x_1) < \infty) = 1$, and the proposition is proved. ■

Another way to state Proposition 1 is to say that for any $x \in K$, the returned process $(\hat{X}_{(xI(\hat{\xi})-t)-},\ 0 \le t \le xI(\hat{\xi}))$, has the same law as $(X_t^{(0)}, 0 \le t < U(x_1))$ given $\Gamma = x$. In [3], the authors show that when the semigroup operator of $X^{(0)}$ is absolutely continuous with respect to the Lebesgue measure with density $p_t(x,y)$, this process is an h-process of $X^{(0)}$, the corresponding harmonic function being $h(x) = \int_0^\infty p_t(x,1)\,dt$.

For $y > 0$, we set

$$\hat{S}_y = \inf\{t : \hat{X}_t \le y\}\,.$$

Corollary 1 *Between the passage times $\hat{S}_{x_n}$ and $\hat{S}_{x_{n+1}}$, the process $\hat{X}$ may be described as follows:*

$$\left(\hat{X}_{\hat{S}(x_n)+t},\ 0 \le t \le \hat{S}_{x_{n+1}} - \hat{S}_{x_n}\right) = \left(\Gamma_n \exp \hat{\xi}^{(n)}_{\hat{\tau}^{(n)}(t/\Gamma_n)},\ 0 \le t \le H_n\right),\quad n \ge 1,$$

where the processes $\hat{\xi}^{(n)}$, $n \ge 1$ are independent between themselves and have the same law as $\hat{\xi}$. Moreover the sequence $(\hat{\xi}^{(n)})$ is independent of Γ defined above and

$$\begin{cases} \hat{\tau}^{(n)}_t = \inf\{s : \int_0^s \exp \hat{\xi}^{(n)}_u\,du \ge t\} \\ H_n = \Gamma_n \int_0^{\hat{T}^{(n)}(\log(x_{n+1}/\Gamma_n))} \exp \hat{\xi}^{(n)}_s\,ds \\ \Gamma_{n+1} = \Gamma_n \exp \hat{\xi}^{(n)}_{\hat{T}^{(n)}(\log x_{n+1}/\Gamma_n)},\ n \ge 1, \quad \Gamma_1 = \Gamma \\ \hat{T}^{(n)}_z = \inf\{t : \hat{\xi}^{(n)}_t \le z\}. \end{cases}$$

For each n, Γ_n is independent of $\xi^{(n)}$ and

$$x_n^{-1}\Gamma_n \stackrel{(d)}{=} x_1^{-1}\Gamma\,. \tag{2.8}$$

Proof: From (2.4) and Proposition 1, the process $\hat{X}$ may be described as

$$\hat{X} = \left(\Gamma \exp \hat{\xi}^{(1)}_{\hat{\tau}^{(1)}(t/\Gamma)},\ 0 \le t \le U(x_1)\right)\,,$$

where $\hat{\xi}^{(1)} \stackrel{(d)}{=} \hat{\xi}$ is independent of $\Gamma = X^{(0)}_{U(x_1)-}$ and $\hat{\tau}^{(1)}_t = \inf\{s : \int_0^s \exp \hat{\xi}^{(1)}_u\,du \ge t\}$. Note that $\Gamma \le x_1$, a.s., so between the passages times $\hat{S}_{x_1} = 0$ and $\hat{S}_{x_2}$, the process $\hat{X}$ is clearly described as in the statement with $\hat{\xi}^{(1)} = \hat{\xi}$ and $\hat{S}_{x_2} - \hat{S}_{x_1} = H_1 = \Gamma \int_0^{\hat{T}^{(1)}(\log(x_2/\Gamma))} \exp \hat{\xi}^{(1)}_s\,ds$.

Now if we set $\hat{\xi}^{(2)} \stackrel{(\mathrm{def})}{=} (\hat{\xi}^{(1)}_{\hat{T}^{(1)}(\log x_2/\Gamma_1)+t} - \hat{\xi}^{(1)}_{\hat{T}^{(1)}(\log x_2/\Gamma_1)},\ t \ge 0)$, then with the definitions of the statement,

$$\begin{aligned} &(\hat{X}_{\hat{S}(x_2)+t},\ t \ge 0) = (\Gamma_2 \exp \hat{\xi}^{(2)}_{\hat{\tau}^{(2)}(t/\Gamma_2)},\ t \ge 0) \quad \text{and} \\ &\hat{S}_{x_3} - \hat{S}_{x_2} = \inf\{t : \hat{X}_{\hat{S}(x_2)+t} \le x_3\} = H_2\,. \end{aligned} \tag{2.9}$$

The process $\hat{\xi}^{(2)}$ is independent of $[(\hat{\xi}^{(1)}_t,\ 0 \le t \le \hat{T}^{(1)}(\log x_2/\Gamma_1)), \Gamma_1]$, hence it is clear that we do not change the law of $\hat{X}$ if, by reconstructing it according to this decomposition, we replace $\hat{\xi}^{(2)}$ by a process with the same law which is independent of $[\hat{\xi}^{(1)}, \Gamma_1]$. Moreover, $\hat{\xi}^{(2)}$ is independent of Γ_2. Relation (2.8) is a consequence of the scaling property. Indeed, we have

$$\left(\frac{x_2}{x_1} X^{(0)}_{tx_1/x_2},\ 0 \le t \le \frac{x_2}{x_1} U(x_1)\right) \stackrel{(d)}{=} \left(X^{(0)}_t,\ 0 \le t \le U(x_2)\right)\,,$$

which implies the identities in law

$$x_1^{-1}X^{(0)}_{U(x_1)-}\stackrel{(d)}{=}x_2^{-1}X^{(0)}_{U(x_2)-}\,,\quad\text{and}\quad x_1^{-1}U(x_1)\stackrel{(d)}{=}x_2^{-1}U(x_2)\,.\tag{2.10}$$

On the other hand, we see from the definition of $\hat{X}$ in Proposition 1 that

$$\left(\hat{X}_{\hat{S}(x_2)+t},\,0\le t\le U(x_1)-\hat{S}(x_2)\right)=\left(X^{(0)}_{(U(x_2)-t)-},\,0\le t\le U(x_2)\right)\,.$$

Then, we obtain (2.8) for $n=2$ from this identity, (2.9) and (2.10). The proof follows by induction. ■

Corollary 2 *With the same notations as in Corollary 1, the time $U(x_n)$ may be decomposed into the sum*

$$U(x_n)=\sum_{k\ge n}\Gamma_k\int_0^{\hat{T}^{(k)}(\log(x_{k+1}/\Gamma_k))}\exp\hat{\xi}^{(k)}_s\,ds\,,\quad a.s.\tag{2.11}$$

In particular, for all $z_n>0$, we have

$$z_n\mathbb{I}_{\{\Gamma_n\ge z_n\}}\int_0^{\hat{T}^{(n)}(\log(x_{n+1}/z_n))}\exp\hat{\xi}^{(n)}_s\,ds\le U(x_n)\le x_nI(\overline{\xi}^{(n)})\,,\quad a.s.,\tag{2.12}$$

where $\overline{\xi}^{(n)}$, $n\ge1$ are Lévy processes with the same law as $\hat{\xi}$.

Proof: Identity (2.11) is a consequence of Corollary 1 and the fact that $U(x_n)=\sum_{k\ge n}\hat{S}_{k+1}-\hat{S}_k$. The first inequality in (2.12) is a consequence of (2.11), which implies: $\Gamma_n\int_0^{\hat{T}^{(n)}(\log(x_{n+1}/\Gamma_n))}\exp\hat{\xi}^{(n)}_s\,ds\le U(x_n)$.

To prove the second inequality in (2.12), it suffices to note that by Proposition 1 and the strong Markov property at time $\hat{S}(x_n)$, for any $n\ge1$, we have the representation

$$\left(\hat{X}_{\hat{S}(x_n)+t},\,0\le t\le U(x_1)-\hat{S}(x_n)\right)=\left(\Gamma_n\exp\overline{\xi}^{(n)}_{\overline{\tau}^{(n)}(t/\Gamma_n)},\,0\le t\le U(x_1)-\hat{S}(x_n)\right)\,,$$

where $\overline{\tau}^{(n)}_t=\inf\{s:\int_0^s\exp\overline{\xi}^{(n)}_u\,du>t\}$ and $\Gamma_n=\hat{X}_{\hat{S}(x_n)}$ (see in Corollary 1) is independent of $\overline{\xi}^{(n)}$ which has the same law as $\hat{\xi}$. It remains to note from (2.5) that $U(x_1)-\hat{S}(x_n)=U(x_n)=\Gamma_nI(\overline{\xi}^{(n)})$ and that $\Gamma_n\le x_n$. ■

To establish our asymptotic results at $+\infty$, we will also need to estimate the law of the time $U(x)$ when x is large. The same reasoning as we did for a sequence which tends to 0 can be done for a sequence which tends to $+\infty$ as we show in the following result.

Corollary 3 *Let (y_n) be an increasing sequence of positive real numbers which tends to $+\infty$. There exists some sequences $(\check{\xi}^{(n)})$, $(\tilde{\xi}^{(n)})$ and $(\check{\Gamma}_n)$, such that for each n, $\check{\xi}^{(n)}\stackrel{(d)}{=}\tilde{\xi}^{(n)}\stackrel{(d)}{=}\hat{\xi}$, $\check{\Gamma}_n\stackrel{(d)}{=}\Gamma$, $\check{\Gamma}_n$ and $\check{\xi}^{(n)}$ are independent; moreover the Lévy processes $(\check{\xi}^{(n)})$ are independent between themselves and we have for all $z_n>0$,*

$$z_n\mathbb{I}_{\{\check{\Gamma}_n\ge z_n\}}\int_0^{\check{T}^{(n)}(\log(y_{n-1}/z_n))}\exp\check{\xi}^{(n)}_s\,ds\le U(y_n)\le y_nI(\tilde{\xi}^{(n)})\,,\quad a.s.\tag{2.13}$$

where $\check{T}^{(n)}_z=\inf\{t:\check{\xi}^{(n)}_t\le z\}$.

Proof: Fix an integer $n \geq 1$ and define the decreasing sequence $x_1, \dots, x_n$ by $x_n = y_1, x_{n-1} = y_2, \dots, x_1 = y_n$, then construct the sequences $\hat{\xi}^{(1)}, \dots, \hat{\xi}^{(n)}$ and $\Gamma_1, \dots, \Gamma_n$ from $x_1, \dots, x_n$ as in Corollary 1 and construct the sequence $\overline{\xi}^{(1)}, \dots, \overline{\xi}^{(n)}$ as in Corollary 2. Now define $\check{\xi}^{(1)} = \hat{\xi}^{(n)}, \check{\xi}^{(2)} = \hat{\xi}^{(n-1)}, \dots, \check{\xi}^{(n)} = \hat{\xi}^{(1)}$ and $\tilde{\xi}^{(1)} = \overline{\xi}^{(n)}, \tilde{\xi}^{(2)} = \overline{\xi}^{(n-1)}, \dots, \tilde{\xi}^{(n)} = \overline{\xi}^{(1)}$ and $\check{\Gamma}_1 = \Gamma_n, \check{\Gamma}_2 = \Gamma_{n-1}, \dots, \check{\Gamma}_n = \Gamma_1$. Then from (2.12), we deduce that for any $k = 2, \dots, n$,

$$z_k \mathrm{I\!I}_{\{\check{\Gamma}_k \geq z_k\}} \int_0^{\check{T}^{(k)}(\log(y_{k-1}/z_k))} \exp \check{\xi}_s^{(k)}\, ds \leq U(y_k) \leq y_k I(\tilde{\xi}^{(k)})\,, \quad a.s.$$

Hence the whole sequences $(\tilde{\xi}^{(n)})$, $(\check{\xi}^{(n)})$ and $(\check{\Gamma}_n)$ are well constructed and fulfill the desired properties. ■

Remark: We emphasize that $\hat{T}^{(n)}(\log(x_{n+1}/\Gamma_n)) = 0$, a.s. on the event $\Gamma_n \leq x_{n+1}$; moreover, we have $\Gamma_n \leq x_n$, a.s., so the first inequality in (2.12) is relevant only when $x_{n+1} < z_n < x_n$. Similarly, in Corollary 2, the first inequality in (2.13) is relevant only when $y_{n-1} < z_n < y_n$.

We end this section with the computation of the law of Γ. Recall that the upward ladder height process (σ_t) associated to ξ is the subordinator which corresponds to the right continuous inverse of the local time at 0 of the reflected process $(\xi_t - \sup_{s \leq t} \xi_s)$, see [1] Chap. V for a proper definition. We denote by ν the Lévy measure of (σ_t).

Lemma 1 *The law of* Γ *is characterized as follows:*

$$\log x_1^{-1}\Gamma \stackrel{(d)}{=} -\mathcal{U}Z\,,$$

where $\mathcal{U}$ *and* Z *are independent r.v.'s,* U *is uniformly distributed over* $[0,1]$ *and the law of* Z *is given by:*

$$\mathbb{P}(Z > u) = \mathbb{E}(\sigma_1)^{-1} \int_{(u,\infty)} s\,\nu(ds), \quad u \geq 0\,. \tag{2.14}$$

In particular, for all $\eta < x_1$, $\mathbb{P}(\Gamma > \eta) > 0$.

Proof. It is proved in [10] that under our hypothesis, (that is $\mathbb{E}(|\hat{\xi}_1|) < +\infty$, $E(\hat{\xi}_1) < 0$ and ξ is not arithmetic), the overshoot process of ξ converges in law, that is

$$\hat{\xi}_{\hat{T}(x)} - x \longrightarrow -\mathcal{U}Z, \quad \text{in law as } x \text{ tends to } -\infty,$$

and the limit law is computed in [8] in terms of the upward ladder height process (σ_t).

On the other hand, we proved in Corollary 1, that

$$\begin{aligned} x_{n+1}^{-1}\Gamma_{n+1} &= \exp[\hat{\xi}^{(n)}_{\hat{T}^{(n)}(\log x_{n+1}/\Gamma_n)} - \log x_{n+1}/\Gamma_n] \stackrel{(d)}{=} x_1^{-1}\Gamma \\ &\stackrel{(d)}{=} \exp[\hat{\xi}_{\hat{T}(\log x_{n+1}/x_n + \log x_1^{-1}\Gamma)} - \log x_{n+1}/x_n - \log x_1^{-1}\Gamma]\,. \end{aligned}$$

Then by taking $x_n = e^{-n^2}$, we deduce from these equalities that $\log x_1^{-1}\Gamma$ has the same law as the limit overshoot of the process $\hat{\xi}$, i.e.

$$\hat{\xi}_{\hat{T}(x)} - x \longrightarrow \log x_1^{-1}\Gamma, \quad \text{in law as } x \text{ tends to } -\infty.$$

$\blacksquare$

As a consequence of the above results we have the following identity in law:

$$U(x) \stackrel{(d)}{=} \frac{x}{x_1}\,\Gamma I(\hat{\xi})\,,$$

(Γ and $I(\hat{\xi})$ being independent) which has been proved in [2], Proposition 3 in the special case where the process $X^{(0)}$ is increasing.

3 The lower envelope

The main result of this section consists in integral tests at 0 and $+\infty$ for the lower envelope of $X^{(0)}$. When no confusion is possible, we set $I \stackrel{\text{(def)}}{=} I(\hat{\xi}) = \int_0^\infty \exp\hat{\xi}_s\,ds$. This theorem means in particular that the asymptotic behaviour of $X^{(0)}$ only depends on the tail behaviour of the law of I, and on this of the law of $\int_0^{\hat{T}_{-q}}\exp\hat{\xi}_s\,ds$, with $\hat{T}_x = \inf\{t : \hat{\xi}_t \le x\}$, for $x \le 0$. So also we set

$$I_q \stackrel{\text{(def)}}{=} \int_0^{\hat{T}_{-q}} \exp\hat{\xi}_s\,ds\,,\quad F(t) \stackrel{\text{(def)}}{=} \mathbb{P}(I > t)\,,\quad F_q(t) \stackrel{\text{(def)}}{=} \mathbb{P}(I_q > t)\,.$$

The following lemma will be used to show that actually, in many particular cases, F suffices to describe the envelope of $X^{(0)}$.

Lemma 2 *Assume that there exists $\gamma > 1$ such that, $\limsup_{t\to+\infty} F(\gamma t)/F(t) < 1$. For any $q > 0$ and $\delta > \gamma e^{-q}$,*

$$\liminf_{t\to+\infty}\frac{F_q((1-\delta)t)}{F(t)} > 0\,.$$

Proof: It follows from the decomposition of ξ into the two independent processes $(\hat{\xi}_s,\, s \le \hat{T}_{-q})$ and $\hat{\xi}' \stackrel{\text{(def)}}{=} (\hat{\xi}_{s+\hat{T}_{-q}} - \hat{\xi}_{\hat{T}_{-q}},\, s \ge 0)$ that

$$I = I_q + e^{\hat{\xi}_{\hat{T}_{-q}}} I' \le I_q + e^{-q} I'$$

where $I' = \int_0^\infty \exp\hat{\xi}'_s\,ds$ is a copy of I which is independent of I_q. Then we can write for any $q > 0$ and $\delta \in (0,1)$, the inequalities

$$\begin{aligned}\mathbb{P}(I > t) &\le \mathbb{P}(I_q + e^{-q}I' \ge t)\\ &\le \mathbb{P}(I_q > (1-\delta)t) + \mathbb{P}(e^{-q}I > \delta t)\,,\end{aligned}$$

so that if moreover, $\delta > \gamma e^{-q}$ then

$$1 - \frac{\mathbb{P}(I > \gamma t)}{\mathbb{P}(I > t)} \le 1 - \frac{\mathbb{P}(I > e^q\delta t)}{\mathbb{P}(I > t)} \le \frac{\mathbb{P}(I_q > (1-\delta)t)}{\mathbb{P}(I > t)}\,.$$

$\blacksquare$

We start by stating the integral test at time 0.

Theorem 1 *The lower envelope of $X^{(0)}$ at 0 is described as follows:*

Let f be an increasing function.

(i) If

$$\int_{0+} F\left(\frac{t}{f(t)}\right)\frac{dt}{t} < \infty\,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}(X_t^{(0)} < (1-\varepsilon)f(t),\, i.o.,\ as\ t \to 0) = 0\,.$$

(ii) If for all $q > 0$,

$$\int_{0+} F_q\left(\frac{t}{f(t)}\right)\frac{dt}{t} = \infty\,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}(X_t^{(0)} < (1+\varepsilon)f(t),\, i.o.,\ as\ t \to 0) = 1\,.$$

(iii) Suppose that $t \mapsto f(t)/t$ is increasing. If there exists $\gamma > 1$ such that,

$$\limsup_{t\to+\infty} \mathbb{P}(I > \gamma t)/\mathbb{P}(I > t) < 1 \ \text{and if}\ \int_{0+} F\left(\frac{t}{f(t)}\right)\frac{dt}{t} = \infty\,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}(X_t^{(0)} < (1+\varepsilon)f(t),\, i.o.,\ as\ t \to 0) = 1\,.$$

Proof: Let (x_n) be a decreasing sequence such that $\lim_n x_n = 0$. Recall the notations of Section 2. We define the events

$$A_n = \{\text{There exists } t \in [U(x_{n+1}), U(x_n)] \text{ such that } X_t^{(0)} < f(t).\}\,.$$

Since $U(x_n)$ tends to 0, a.s. when n goes to $+\infty$, we have:

$$\{X_t^{(0)} < f(t), \text{i.o., as } t \to 0\} = \limsup_n A_n\,. \tag{3.1}$$

Since f is increasing, the following inclusions hold:

$$\{x_n \le f(U(x_n))\} \subset A_n \subset \{x_{n+1} \le f(U(x_n))\}\,. \tag{3.2}$$

Then we prove the convergent part (i). Let us choose $x_n = r^{-n}$ for $r > 1$, and recall from relation (2.12) above that $U(r^{-n}) \le r^{-n} I(\overline{\xi}^{(n)})$. From this inequality and (3.2), we can write:

$$A_n \subset \{r^{-(n+1)} \le f(r^{-n} I(\overline{\xi}^{(n)}))\}\,. \tag{3.3}$$

Let us denote $I(\hat{\xi})$ simply by I. From Borel Cantelli's Lemma, (3.3) and (3.1),

$$\text{if } \textstyle\sum_n \mathbb{P}(r^{-(n+1)} \le f(r^{-n} I)) < \infty \text{ then } \mathbb{P}(X_t^{(0)} < f(t), \text{i.o., as } t \to 0) = 0. \tag{3.4}$$

Note that $\int_1^{+\infty}\mathbb{P}(r^{-t}\le f(r^{-t}I))\,dt=\int_{0+}^{+\infty}\mathbb{P}(s<f(s)I,\,s<I/r)/(s\log r)\,ds$, hence since f is increasing, we have the inequalities:

$$\sum_{n=1}^{\infty}\mathbb{P}(r^{-n}\le f(r^{-(n+1)}I))\le\int_{0+}^{+\infty}\mathbb{P}\left(\frac{s}{f(s)}<I,\,s<\frac{I}{r}\right)\frac{ds}{s\log r}\le\sum_{n=1}^{\infty}\mathbb{P}(r^{-(n+1)}\le f(r^{-n}I))\,. \tag{3.5}$$

With no loss of generality, we can restrict ourself to the case $f(0)=0$, so it is not difficult to check that for any $r>1$,

$$\int_{0+}\mathbb{P}\left(\frac{s}{f(s)}<I,\,s<\frac{I}{r}\right)\frac{ds}{s}<+\infty,\quad\text{if and only if}\quad\int_{0+}\mathbb{P}\left(\frac{s}{f(s)}<I\right)\frac{ds}{s}<+\infty\,. \tag{3.6}$$

Suppose the latter condition holds, then from (3.5), for all $r>1$, $\sum_{n=2}^{\infty}\mathbb{P}(r^{-(n+1)}\le r^{-2}f(r^{-n}I))<+\infty$ and from (3.4), for all $r>1$, $\mathbb{P}(X_t^{(0)}<r^{-2}f(t),\text{i.o., as }t\to 0)=0$ which proves the desired result.

Now we prove the divergent part (ii). Again, we choose $x_n=r^{-n}$ for $r>1$, and $z_n=kr^{-n}$, where $k=1-\varepsilon+\varepsilon/r$ and $0<\varepsilon<1$, (so that $x_{n+1}<z_n<x_n$). We set

$$B_n=\{r^{-n}\le f_{r,\varepsilon}(kr^{-n}\mathbb{1}_{\{\Gamma_n\ge kr^{-n}\}}I^{(n)})\}\,,$$

where, $f_{r,\varepsilon}(t)=rf(t/k)$ and with the same notations as in Corollary 2, for each n,

$$I^{(n)}\stackrel{(\mathrm{def})}{=}\int_0^{\hat{T}^{(n)}(\log(x_{n+1}/z_n))}\exp\hat{\xi}_s^{(n)}\,ds\stackrel{(d)}{=}\int_0^{\hat{T}(\log(1/rk))}\exp\hat{\xi}_s\,ds \tag{3.7}$$

is independent of Γ_n, and Γ_n is such that $x_n^{-1}\Gamma_n\stackrel{(d)}{=}x_1^{-1}\Gamma$. Moreover the r.v.'s $I^{(n)}$, $n\ge 1$ are independent between themselves and identity (3.7) shows that they have the same law as I_q defined in Lemma 2, where $q=-\log(1/rk)$. With no loss of generality, we may assume that $f(0)=0$, so that we can write $B_n=\{r^{-n}\le f_{r,\varepsilon}(kr^{-n}I^{(n)}),\,\Gamma_n\ge kr^{-n}\}$ and from the above arguments we deduce

$$\mathbb{P}(B_n)=\mathbb{P}(r^{-n}\le f_{r,\varepsilon}(kr^{-n}I_q))\mathbb{P}(\Gamma\ge kr^{-1})\,. \tag{3.8}$$

The arguments which are developed above to show (3.5) and (3.6), are also valid if we replace I by I_q. Hence from the hypothesis, since $\int_{0+}\mathbb{P}(s<f(s)I_q)\,ds/s=+\infty$, then from (3.5) and (3.6) applied to I_q, we have $\sum_{n=1}^{\infty}\mathbb{P}(r^{-(n+1)}\le f(r^{-n}I_q))=\sum_{n=1}^{\infty}\mathbb{P}(r^{-n}\le f_{r,\varepsilon}(kr^{-n}I_q))=\infty$, and from (3.8) we have $\sum_n\mathbb{P}(B_n)=+\infty$. Then another application of (3.8), gives for any n and m,

$$\begin{aligned}\mathbb{P}(B_n\cap B_m)&\le\mathbb{P}(r^{-n}\le f_{r,\varepsilon}(kr^{-n}I_q))\mathbb{P}(r^{-m}\le f_{r,\varepsilon}(kr^{-m}I_q))\\ \mathbb{P}(B_n\cap B_m)&\le\mathbb{P}(\Gamma\ge kr^{-1})^{-2}\mathbb{P}(B_n)\mathbb{P}(B_m)\,,\end{aligned}$$

where $\mathbb{P}(\Gamma\ge kr^{-1})>0$, from (2.14). Hence from the extension of Borel-Cantelli's lemma given in [13],

$$\mathbb{P}(\limsup B_n)\ge\mathbb{P}(\Gamma\ge kr^{-1})^2>0\,. \tag{3.9}$$

Then recall from Corollary 2 the inequality $kr^{-n}\mathbb{1}_{\{\Gamma_n\ge kr^{-n}\}}I^{(n)}\le U(r^{-n})$ which implies from (3.2) that $B_n\subset A_n$, (where in the definition of A_n we replaced f by $f_{r,\varepsilon}$). So,

from (3.9), $\mathbb{P}(\limsup_n A_n) > 0$, but since $X^{(0)}$ is a Feller process and since $\limsup_n A_n$ is a tail event, we have $\mathbb{P}(\limsup_n A_n) = 1$. We deduce from the scaling property of $X^{(0)}$ and (3.1) that

$$\begin{aligned} \mathbb{P}(X_t^{(0)} \le f_{r,\varepsilon}(t), \text{i.o., as } t \to 0.) &= \mathbb{P}(X_{kt}^{(0)} \le rf(t), \text{i.o., as } t \to 0.) \\ &= \mathbb{P}(X_t^{(0)} \le k^{-1}rf(t), \text{i.o., as } t \to 0.) = 1\,. \end{aligned}$$

Since $k = 1 - \varepsilon + \varepsilon/r$, with $r > 1$ and $0 < \varepsilon < 1$ arbitrary chosen, we obtain (ii).

Now we prove the divergent part (iii). The sequences (x_n) and (z_n) are defined as in the proof of (ii) above. Recall that $q = -\log(1/rk)$ and take $\delta > \gamma e^{-q}$ as in Lemma 2. With no loss of generality, we may assume that $f(t)/t \to 0$, as $t \to 0$. Then from the hypothesis in (iii) and Lemma 2, we have

$$\int_{0+} F_q\left(\frac{(1-\delta)t}{f(t)}\right)\frac{dt}{t} = \infty\,.$$

As already noticed above, this is equivalent to $\int_1^{+\infty} \mathbb{P}((1-\delta)r^{-t} \le f(r^{-t}I_q))\,dt = \infty$. Since $t \mapsto f(t)/t$ increases, $\int_1^{+\infty} \mathbb{P}((1-\delta)r^{-t} \le f(r^{-t}I_q))\,dt \le \sum_1^\infty \mathbb{P}((1-\delta)r^{-n} \le f(r^{-n}I_q)) = \infty$. Set $f_r^{(\delta)}(t) = (1-\delta)^{-1}f(t/k)$, then

$$\sum_1^\infty \mathbb{P}(r^{-n} \le f_r^{(\delta)}(kr^{-n}I_q)) = \infty\,.$$

Similarly as in the proof of (ii), define $B'_n = \{r^{-n} \le f_r^{(\delta)}(kr^{-n}I^{(n)}),\, \Gamma_n \ge kr^{-n}\}$. Then $B'_n \subset A_n$, (where in the definition of A_n we replaced f by $f_r^{(\delta)}$). From the same arguments as above, since $\sum^\infty \mathbb{P}(B'_n) = \infty$, we have $\mathbb{P}(\limsup_n A_n) = 1$, hence from the scaling property of $X^{(0)}$ and (3.1)

$$\begin{aligned} \mathbb{P}(X_t^{(0)} \le f_r^{(\delta)}(t), \text{i.o., as } t \to 0.) &= \mathbb{P}(X_{kt}^{(0)} \le (1-\delta)^{-1}f(t), \text{i.o., as } t \to 0.) \\ &= \mathbb{P}(X_t^{(0)} \le k^{-1}(1-\delta)^{-1}f(t), \text{i.o., as } t \to 0.) = 1\,. \end{aligned}$$

Since $k = 1 - \varepsilon + \varepsilon/r$, with $r > 1$ and $0 < \varepsilon < 1$ and $\delta > \gamma e^{-q} = \gamma/(r + \varepsilon(1-r))$, by choosing r sufficiently large and ε sufficiently small, δ can be taken sufficiently small so that $k^{-1}(1-\delta)^{-1}$ is arbitrary close to 1. ■

The divergent part of the integral test at $+\infty$ requires the following Lemma.

Lemma 3 *For any Lévy process ξ such that $0 < \mathbb{E}(\xi_1) \le \mathbb{E}(|\xi_1|) < \infty$, and for any $q \ge 0$,*

$$\mathbb{E}\left(\left|\inf_{t \le T_q} \xi_t\right|\right) < \infty\,,$$

where $T_q = \inf\{t : \xi_t \ge q\}$.

Proof. The proof bears upon a result on stochastic bounds for Lévy processes due to Doney [9] which we briefly recall. Let ν_n be the time at which the n-th jump of ξ whose value lies in $[-1,1]^c$, occurs and define

$$I_n = \inf_{\nu_n \le t < \nu_{n+1}} \xi_t\,.$$

Theorem 1.1 in [9] asserts that the sequence (I_n) admits the representation

$$I_n = S_n^{(-)} + \tilde{\imath}_0, \quad n \geq 0\,,$$

where $S^{(-)}$ is a random walk with the same distribution as $(\xi(\nu_n),\, n \geq 0)$ and $\tilde{\imath}_0$ is independent of $S^{(-)}$. For $a \geq 0$, let $\sigma(a) = \min\{n : S_n^{(-)} > a\}$, then for any $q \geq 0$, we have the inequality

$$\min_{n \leq \sigma(q+|\tilde{\imath}_0|)} (S_n^{(-)} + \tilde{\imath}_0) \leq \inf_{t \leq T_q} \xi_t\,. \tag{3.10}$$

On the other hand, it follows from our hypothesis on ξ that $0 < \mathbb{E}(S_1^{(-)}) \leq \mathbb{E}(|S_1^{(-)}|) < +\infty$, hence from Theorem 2 of [12] and its proof, there exists a finite constant C which depends only on the law of $S^{(-)}$ such that for any $a \geq 0$,

$$\mathbb{E}\left(\left|\min\nolimits_{n \leq \sigma(a)} S_n^{(-)}\right|\right) \leq C\mathbb{E}(\sigma(a))\mathbb{E}(|S_1^{(-)}|)\,. \tag{3.11}$$

Moreover from (1.5) in [12], there are finite constants A and B depending only on the law of $S^{(-)}$ such that for any $a \geq 0$

$$\mathbb{E}(\sigma(a)) \leq A + Ba\,. \tag{3.12}$$

Since $\tilde{\imath}_0$ is integrable (see [9]), the result follows from (3.10), (3.11), (3.12) and the independence between $\tilde{\imath}_0$ and $S^{(-)}$. ■

Theorem 2 *The lower envelope of $X^{(x)}$ at $+\infty$ is described as follows:*

Let f be an increasing function.

(*i*) *If*

$$\int^{+\infty} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} < \infty\,,$$

then for all $\varepsilon > 0$, and for all $x \geq 0$,

$$\mathbb{P}(X_t^{(x)} < (1-\varepsilon) f(t), \textit{i.o., as } t \to +\infty) = 0\,.$$

(*ii*) *If for all $q > 0$,*

$$\int^{+\infty} F_q\left(\frac{t}{f(t)}\right) \frac{dt}{t} = \infty\,,$$

then for all $\varepsilon > 0$, and for all $x \geq 0$,

$$\mathbb{P}(X_t^{(x)} < (1+\varepsilon) f(t), \textit{i.o., as } t \to +\infty) = 1\,.$$

(*iii*) *Assume that there exists $\gamma > 1$ such that, $\limsup_{t\to+\infty} \mathbb{P}(I > \gamma t)/\mathbb{P}(I > t) < 1$. Assume also that $t \mapsto f(t)/t$ is decreasing. If*

$$\int^{+\infty} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} = \infty\,,$$

then for all $\varepsilon > 0$, and for all $x \geq 0$,

$$\mathbb{P}(X_t^{(x)} < (1+\varepsilon) f(t), \textit{i.o., as } t \to +\infty) = 1\,.$$

Proof: We first consider the case where $x = 0$. The proof is very similar to this of Theorem 1. We can follow the proofs of (i), (ii) and (iii) line by line, replacing the sequences $x_n = r^{-n}$ and $z_n = kr^{-n}$ respectively by the sequences $x_n = r^n$ and $z_n = kr^n$, and replacing Corollary 2 by Corollary 3. Then with the definition

$$A_n = \{\text{There exists } t \in [U(r^n), U(r^{n+1})] \text{ such that } X_t^{(0)} < f(t).\}\,,$$

we see that the event $\limsup A_n$ belongs to the tail sigma-field $\cap_t \sigma\{X_s^{(0)} : s \geq t\}$ which is trivial from the representation (1.2) and the Markov property.

The only thing which has to be checked more carefully is the counterpart at $+\infty$ of the equivalence (3.6). Indeed, since in that case $\int_1^\infty \mathbb{P}(rt < f(r^t I)\, dt = \int_{0+}^\infty \mathbb{P}(s/f(s) < I_q,\, s > rI_q)\, ds/(s \log r)$, in the proof of (ii) and (iii), we need to make sure that for any $r > 1$,

$$\int^{+\infty} \mathbb{P}\left(\frac{s}{f(s)} < I_q\right) \frac{ds}{s} = +\infty \quad \text{implies} \quad \int^{+\infty} \mathbb{P}\left(\frac{s}{f(s)} < I_q < sr\right) \frac{ds}{s} = +\infty\,. \tag{3.13}$$

To this aim, note that

$$\int_1^\infty \mathbb{P}\left(\frac{s}{f(s)} < I_q < sr\right) \frac{ds}{s} = \int_1^\infty \mathbb{P}\left(\frac{s}{f(s)} < I_q\right) - \mathbb{P}\left(\frac{s}{f(s)} < I_q,\, sr < I_q\right) \frac{ds}{s}\,,$$

and since f is increasing, we have

$$\int_1^\infty \mathbb{P}\left(\frac{s}{f(s)} < I_q,\, sr < I_q\right) \frac{ds}{s} < +\infty \quad \text{if and only if} \quad \int_1^\infty \mathbb{P}\left(s < I_q\right) \frac{ds}{s} < +\infty\,.$$

But

$$\int_1^\infty \mathbb{P}\left(s < I_q\right) \frac{ds}{s} = \mathbb{E}(\log^+ I_q)\,.$$

Note that from our hypothesis on ξ, we have $\mathbb{E}(\hat{T}_{-q}) < +\infty$, then the conclusion follows from the inequality

$$\mathbb{E}(\log^+ I_q) \leq \mathbb{E}\left(\sup_{0 \leq s \leq \hat{T}_{-q}} \hat{\xi}_s\right) + \mathbb{E}(\hat{T}_{-q})$$

and Lemma 3. This achieves the proof of the theorem for $x = 0$.

Now we prove (i) for any $x > 0$. Let f be an increasing function such that $\int^{+\infty} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} < +\infty$. Let $x > 0$, put $S_x = \inf\{t : X_t^{(0)} \geq x\}$ and denote by μ_x the law of $X_{S_x}^{(0)}$. From the Markov property at time S_x, we have for all $\varepsilon > 0$,

$$\begin{aligned}
& \mathbb{P}(X_t^{(0)} < (1-\varepsilon) f(t - S_x), \text{i.o., as } t \to +\infty) \\
= & \int_{[x,\infty)} \mathbb{P}(X_t^{(y)} < (1-\varepsilon) f(t), \text{i.o., as } t \to +\infty)\, \mu_x(dy) \\
\leq & \ \mathbb{P}(X_t^{(0)} < (1-\varepsilon) f(t), \text{i.o., as } t \to +\infty) = 0\,.
\end{aligned} \tag{3.14}$$

If x is an atom of μ_x, then the inequality (3.14) shows that

$$\mathbb{P}(X_t^{(x)} < (1-\varepsilon) f(t), \text{i.o., as } t \to +\infty) = 0$$

and the result is proved. Suppose that x is not an atom of μ_x. Recall from section 1 that $\log(x_1^{-1}\Gamma)$ is the limit in law of the overshoot process $\hat{\xi}_{\hat{T}_z} - z$, as $z \to +\infty$. Moreover, it follows from [5], Theorem 1 that $X^{(0)}_{S_x} \stackrel{(d)}{=} \frac{xx_1}{\Gamma}$. Hence, from Lemma 1, we have for any $\eta > 0$, $\mu_x(x, x+\eta) > 0$. Then, the inequality (3.14) implies that for any $\eta > 0$, there exists $y \in (x, x+\eta)$ such that $\mathbb{P}(X_t^{(y)} < (1-\varepsilon)f(t), \text{i.o., as } t \to +\infty) = 0$, for all $\varepsilon > 0$. It allows us to conclude.

Parts (*ii*) and (*iii*) can be proved through the same way. ■

We recall that to obtain these tests for any scaling index $\alpha > 0$, it suffices to consider the process $(X^{(0)})^{1/\alpha}$ in the above theorems. The same remark holds for the results of the next sections.

4 The regular case

The first type of tail behaviour of I that we consider is the case where F is regularly varying at infinity, i.e.

$$F(t) \sim \lambda t^{-\gamma} L(t)\,, \quad t \to +\infty\,, \tag{4.1}$$

where $\gamma > 0$ and L is a slowly varying function at $+\infty$. As shown in the following lemma, under this assumption, for any $q > 0$ the functions F_q and F are equivalent, i.e. $F_q \asymp F$.

Lemma 4 *Recall that $I_q = \int_0^{T_{-q}} \exp \hat{\xi}_s\, ds$ and $F_q(t) = \mathbb{P}(I_q > t)$. If (4.1) holds then for all $q > 0$,*

$$(1 - e^{-\gamma q})F(t) \le F_q(t) \le F(t)\,, \tag{4.2}$$

for all t large enough.

Proof: Recall from Lemma 2, that if $(\hat{\xi}_s,\, s \le \hat{T}_{-q})$ and $\hat{\xi}' \stackrel{(\mathrm{def})}{=} (\hat{\xi}_{s+\hat{T}_{-q}} - \hat{\xi}_{\hat{T}_{-q}},\, s \ge 0)$ then

$$I = I_q + \exp(\hat{\xi}_{\hat{T}_{-q}})I' \le I_q + e^{-q} I' \tag{4.3}$$

where $I' = \int_0^\infty \exp \hat{\xi}'_s\, ds$ is a copy of I which is independent of I_q. It yields the second equality of the lemma. To show the first inequality, we write for all $\delta > 0$,

$$\begin{aligned}
\mathbb{P}(I > (1+\delta)t) &\le \mathbb{P}(I_q + e^{-q}I' \ge (1+\delta)t) \\
&\le \mathbb{P}(I_q > t) + \mathbb{P}(e^{-q}I > t) + \mathbb{P}(I_q > \delta t)\mathbb{P}(e^{-q}I > \delta t) \\
&\le \mathbb{P}(I_q > t) + \mathbb{P}(e^{-q}I > t) + \mathbb{P}(I > \delta t)\mathbb{P}(e^{-q}I > \delta t)\,,
\end{aligned}$$

so that

$$\liminf_{t \to +\infty} \frac{\mathbb{P}(I_q > t)}{\mathbb{P}(I > t)} \ge (1+\delta)^{-\gamma} - e^{-q\gamma}\,,$$

and the result follows since δ can be chosen arbitrary small. ■

The regularity of the behaviour of F allows us to drop the ε of Theorems 1 and 2 in the next integral test.

Theorem 3 *Under condition (4.1), the lower envelope of $X^{(0)}$ at 0 and at $+\infty$ is as follows:*

Let f be an increasing function, such that either $\lim_{t\downarrow 0} f(t)/t = 0$, or $\liminf_{t\downarrow 0} f(t)/t > 0$, then:

$$\mathbb{P}(X_t^{(0)} < f(t), \textit{i.o., as } t \to 0) = \begin{cases} 0 \\ 1 \end{cases},$$

according as

$$\int_{0+} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Let g be an increasing function, such that either $\lim_{t\uparrow+\infty} g(t)/t = 0$, or $\liminf_{t\uparrow+\infty} g(t)/t > 0$, then for all $x \ge 0$,

$$\mathbb{P}(X_t^{(x)} < g(t), \textit{i.o., as } t \to +\infty) = \begin{cases} 0 \\ 1 \end{cases},$$

according as

$$\int^{+\infty} F\left(\frac{t}{g(t)}\right) \frac{dt}{t} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Proof: First let us check that for any constant $\beta > 0$:

$$\int_{0+}^{\lambda} F\left(\frac{s}{f(s)}\right) \frac{ds}{s} < \infty \quad \text{if and only if} \quad \int_{0+}^{\lambda} F\left(\frac{\beta s}{f(s)}\right) \frac{ds}{s} < \infty\,. \tag{4.4}$$

From the hypothesis, either $\lim_{s\downarrow 0} f(s)/s = 0$, or $\liminf_{s\downarrow 0} f(s)/s > 0$. In the first case, we deduce (4.4) from (4.1). In the second case, since for any $0 < \lambda < \infty$, $\mathbb{P}(I > \lambda) > 0$, and $\limsup_{u\downarrow 0} u/f(u) < +\infty$, we have for any s, $0 < \mathbb{P}\left(\limsup_{u\downarrow 0} \frac{u}{f(u)} < I\right) < \mathbb{P}\left(\frac{s}{f(s)} < I\right)$ so both of the integrals above are infinite.

Now it follows from Theorem 1 that if $\int_{0+} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} < \infty$ then for all $\varepsilon > 0$, $\mathbb{P}(X_t^{(0)} < (1-\varepsilon)f(t), \text{i.o., as } t \to 0) = 0$. If $\int_{0+} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} = \infty$ then from Lemma 4, for all $q > 0$, $\int_{0+} F_q\left(\frac{t}{f(t)}\right) \frac{dt}{t} = \infty$, and it follows from Theorem 1 (*ii*) that for all $\varepsilon > 0$, $\mathbb{P}(X_t^{(0)} < (1+\varepsilon)f(t), \text{i.o., as } t \to 0) = 1$. Then the equivalence (4.4) allows us to drop ε in these implications.

The test at $+\infty$ is proven through the same way. ■

Remarks:

1. It is possible to obtain the divergent parts of Theorem 3 by applying parts (*iii*) of Theorems 1 and 2 but then, one has to assume that $f(t)/t$ is an increasing (respectively a decreasing) function for the test at 0 (respectively at $+\infty$), which is slightly stronger than the hypothesis on f of Theorem 3.

2. This result is due to Dvoretzky and Erdös [11] and Motoo [17] when $X^{(0)}$ is a transient Bessel process, i.e. the square root of the solution of the SDE:

$$Z_t = 2\int_0^t \sqrt{Z_s}\, dB_s + \delta t\,, \tag{4.5}$$

where $\delta > 2$ and B is a standard Brownian motion. (Recall that when δ is an integer, $X^{(0)} = \sqrt{Z}$ has the same law as the norm of the δ-dimensional Brownian motion.) Processes $X^{(0)} = \sqrt{Z}$ such that Z satisfies the equation (4.5) with $\delta > 2$ are the only continuous self-similar Markov process with index $\alpha = 2$, which drifts towards $+\infty$. In this particular case, thanks to the time-inversion property, i.e.:

$$(X_t,\, t > 0) \stackrel{(d)}{=} (tX_{1/t},\, t > 0),$$

we may deduce the test at $+\infty$ from the test at 0.
3. A possible way to improve the test at ∞ in the general case (that is in the setting of Theorem 1) would be to first establish it for the Ornstein-Uhlenbeck process associated to $X^{(0)}$, i.e. $(e^{-t}X^{(0)}(e^t),\, t \geq 0)$, as Motoo did for Bessel processes in [17]. This would allow us to consider test functions which are not necessarily increasing.

Examples:
1. Examples of such behaviours are given by transient Bessel processes raised to any power and more generally when the process ξ satisfies the so called Cramer's condition, that is,

$$\text{there exists } \gamma > 0 \text{ such that } E(e^{-\gamma\xi_1}) = 1. \tag{4.6}$$

In that case, Rivero [19] and Maulik and Zwart [16] proved by using results of Kesten and Goldie on tails of solutions of random equations that the behavior of $P(I > t)$ is given by

$$F(t) \sim Ct^{-\gamma}\,, \quad \text{as } t \to +\infty, \tag{4.7}$$

where the constant C is explicitly computed in [18] and [16].

2. Stable Lévy processes conditioned to stay positive are themselves positive self-similar Markov processes which belong to the regular case. These processes are defined as h-processes of the initial process when it starts from $x > 0$ and killed at its first exit time of $(0, \infty)$. Denote by (q_t) the semigroup of a stable Lévy process Y with index $\alpha \in (0, 2]$, killed at time $R = \inf\{t : Y_t \leq 0\}$. The function $h(x) = x^{\alpha(1-\rho)}$, where $\rho = \mathbb{P}(Y_1 \geq 0)$, is invariant for the semi-group (q_t), i.e. for all $x \geq 0$ and $t \geq 0$, $E_x(h(Y_t)\mathbb{1}_{\{t<R\}}) = h(x)$, ($E_x$ denotes the law of $Y + x$). The Lévy process Y conditioned to stay positive is the strong Markov process whose semigroup is

$$p_t^{\uparrow}(x, dy) := \frac{h(y)}{h(x)} q_t(x, dy),\ x > 0, y > 0,\ t \geq 0\,. \tag{4.8}$$

We will denote this process by $X^{(x)}$ when it is issued from $x > 0$. We refer to [6] for more on the definition of Lévy processes conditioned to stay positive and for a proof of the above facts. It is easy to check that the process $X^{(x)}$ is self-similar and drifts towards $+\infty$. Moreover, it is proved in [6], Theorem 6 that $X^{(x)}$ converges weakly as $x \to 0$ towards a non degenerated process $X^{(0)}$ in the Skorohod's space, so from [5], the underlying Lévy process in the Lamperti representation of $X^{(x)}$ satisfies condition (H).

We can check that the law of $X^{(x)}$ belongs to the regular case by using the equality in law (2.4). Indeed, it follows from Proposition 1 and Theorem 4 in [6] that the law of the exponential functional I is given by

$$\mathbb{P}(t < x^\alpha I) = x^{1-\alpha\rho} E_{-x}(\hat{Y}_t^{\alpha\rho-1}\mathbb{1}_{\{t<\hat{R}\}})\,, \tag{4.9}$$

where $\hat{Y} = -Y$ and $\hat{R} = \inf\{t : \hat{Y}_t \le 0\}$. If Y (and thus $X^{(0)}$) has no positive jumps, then $\alpha\rho = 1$ and it follows from (4.9) and Lemma 1 in [7] that

$$\mathbb{P}(t < I) = Ct^{-\rho}\,. \tag{4.10}$$

We conjecture that (4.10) is also valid when Y has positive jumps. We also emphasize the possibility that the underlying Lévy process in the Lamperti representation of $X^{(x)}$ even satisfies (4.6) with $\gamma = \rho$.

5 The log regular case

The second type of behaviour that we shall consider is when $\log F$ is regularly varying at $+\infty$, i.e.

$$-\log F(t) \sim \lambda t^{\beta} L(t)\,, \quad \text{as } t \to \infty, \tag{5.1}$$

where $\lambda > 0$, $\beta > 0$ and L is a function which varies slowly at $+\infty$. Define the function ψ by

$$\psi(t) \overset{\text{(def)}}{=} \frac{t}{\inf\{s : 1/F(s) > |\log t|\}}\,, \quad t > 0\,,\ t \neq 1\,. \tag{5.2}$$

Then the lower envelope of $X^{(0)}$ may be described as follows:

Theorem 4 *Under condition (5.1), the process $X^{(0)}$ satisfies the following law of the iterated logarithm:*

(i)

$$\liminf_{t\to 0} \frac{X_t^{(0)}}{\psi(t)} = 1\,, \quad \textit{almost surely.} \tag{5.3}$$

(ii) For all $x \ge 0$,

$$\liminf_{t\to +\infty} \frac{X_t^{(x)}}{\psi(t)} = 1\,, \quad \textit{almost surely.} \tag{5.4}$$

Proof: We shall apply Theorem 1. We first have to check that under hypothesis (5.1), the conditions of part *(iii)* in Theorem 1 are satisfied. On the one hand, from (5.1) we deduce that for any $\gamma > 1$, $\limsup F(\gamma t)/F(t) = 0$. On the other hand, it is easy to see that both $\psi(t)$ and $\psi(t)/t$ are increasing in a neighbourhood of 0.

Let $\overline{L}$ be a slowly varying function such that

$$-\log F(\lambda^{-1/\beta} t^{1/\beta} \overline{L}(t)) \sim t\,, \quad \text{as } t \to +\infty. \tag{5.5}$$

Th. 1.5.12, p.28 in [4] ensures that such a function exists and that

$$\inf\{s : -\log F(s) > t\} \sim \lambda^{-1/\beta} t^{1/\beta} \overline{L}(t)\,, \quad \text{as } t \to +\infty. \tag{5.6}$$

Then we have for all $k_1 < 1$ and $k_2 > 1$ and for all t sufficiently large,

$$k_1 \lambda^{-1/\beta} t^{1/\beta} \overline{L}(t) \le \inf\{s : -\log F(s) > t\} \le k_2 \lambda^{-1/\beta} t^{1/\beta} \overline{L}(t)$$

so that for ψ defined above and for all $k_2'>0$,

$$-\log F\left(\frac{t\,k_2'}{k_2\psi(t)}\right)\le -\log F(k_2'\lambda^{-1/\beta}(\log|\log t|)^{1/\beta}\overline{L}(\log|\log t|)) \tag{5.7}$$

for all t sufficiently small. But from (5.5), for all $k_2''>1$ and for all t sufficiently small,

$$-\log F(k_2'\lambda^{-1/\beta}(\log|\log t|)^{1/\beta}\overline{L}(\log|\log t|))\le k_2''{k_2'}^{\beta}\log|\log t|\,,$$

hence

$$F\left(\frac{t\,k_2'}{k_2\psi(t)}\right)\ge(|\log t|)^{-k_2''{k_2'}^{\beta}}\,.$$

By choosing $k_2'<1$ and $k_2''<(k_2')^{-\beta}$, we obtain the convergence of the integral

$$\int_{0+}F\left(\frac{t\,k_2'}{k_2\psi(t)}\right)\frac{dt}{t}\,,$$

for all $k_2>1$ and $k_2'<1$, which proves that for all $\varepsilon>0$,

$$\mathbb{P}(X_t^{(0)}<(1+\varepsilon)\psi(t),\text{i.o., as }t\to 0)=1$$

from Theorem 1 (*iii*). The convergent part is proven through the same way so that from Threorem 1 (*i*), one has for all $\varepsilon>0$,

$$\mathbb{P}(X_t^{(0)}<(1-\varepsilon)\psi(t),\text{i.o., as }t\to 0)=0$$

and the conclusion follows.

Condition (5.1) implies that $\psi(t)$ is increasing in a neighbourhood of $+\infty$ whereas $\psi(t)/t$ is decreasing in a neighbourhood of $+\infty$. Hence, the proof of the result at $+\infty$ is done through the same way as at 0, by using Theorem 2, (*i*) and (*iii*). ∎

Example:
An example of such a behaviour is provided by the case where the process $X^{(0)}$ is increasing, that is when the underlying Lévy process ξ is a subordinator. Then Rivero [18], see also Maulik and Zwart [16] proved that when the Laplace exponent ϕ of ξ which is defined by

$$\exp(-t\phi(\lambda))=\mathbb{E}(\exp(\lambda\hat{\xi}_t))\,,\quad \lambda>0,\;\; t\ge 0$$

is regularly varying at $+\infty$ with index $\beta\in(0,1)$, the upper tail of the law of I and the asymptotic behavior of ϕ at $+\infty$ are related as follows:

Proposition 2 *Suppose that ξ is a subordinator whose Laplace exponent ϕ varies regularly at infinity with index $\beta\in(0,1)$, then*

$$-\log F(t)\sim(1-\beta)\phi^{\leftarrow}(t)\,,\quad \textit{as } t\to\infty,$$

where $\phi^{\leftarrow}(t)=\inf\{s>0:s/\phi(s)>t\}$.

Then by using an argument based on the study of the associated Ornstein-Uhlenbeck process $(e^{-t}X^{(0)}(e^t),\, t \geq 0)$ Rivero [18] derived from Proposition 2 the following result. Define

$$\varphi(t) = \frac{\phi(\log|\log t|)}{\log|\log t|}, \quad t > e\,.$$

Corollary 4 *If ϕ is regularly varying at infinity with index $\beta \in (0,1)$ then*

$$\liminf_{t\downarrow 0} \frac{X^{(0)}}{t\varphi(t)} = (1-\beta)^{1-\beta} \quad \textit{and} \quad \liminf_{t\uparrow+\infty} \frac{X^{(0)}}{t\varphi(t)} = (1-\beta)^{1-\beta}, \quad \textit{a.s.}$$

This corollary is also a consequence of Proposition 2 and Theorem 4. To establish Corollary 4, Rivero assumed moreover that the density of the law of the exponential functional I is decreasing and bounded in a neighbourhood of $+\infty$. This additional assumption is actually needed to establish an integral test which involves the density of I and which implies Corollary 4.

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